JOURNAL OF APPROXIMATION THEORY 12, 435–436 (1974)

Note

On the Degree of Approximation by Step Functions

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Let $-\infty < a < b < \infty$. For $n = 1, 2,...$ set $x_j^{(n)} = a + j(b - a)\, n^{-1}$, $j = 0, 1,..., n$, so that $I_{n,j} = (x_{j-1}^{(n)}, x_j^{(n)})$, $j = 1, 2,..., n$, are n congruent subintervals of (a, b). Also, for $n = 1, 2,...$ let S_n be the set of all real functions on $[a, b] - \{x_0^{(n)}, x_1^{(n)},..., x_n^{(n)}\}$ which are constant in each $I_{n,j}$, $j = 1, 2,..., n$.

THEOREM. *Given a real function f on (a, b), and $\alpha > 0$, a necessary and sufficient condition for f to satisfy in (a, b) a Lipschitz condition of order α is that for $n = 1, 2,...$ there exists an $s_n \in S_n$ such that*

$$\sup_{x\in[a,b]-\{x_0^{(n)},x_1^{(n)},\ldots,x_n^{(n)}\}} | f(x) - s_n(x)| \leqslant C/n^\alpha, \tag{1}$$

C being a constant.

Necessity is immediate: Suppose $|f(y) - f(x)| \leqslant L(y - x)^\alpha$ whenever $a < x < y < b$. For $n = 1, 2,...$ let $s_n(x)$ be, throughout each $I_{n,j}$, $j = 1, 2,..., n$, the value of f at the midpoint of $I_{n,j}$ so that, throughout that interval, $|f(x) - s_n(x)| \leqslant L[(b - a)/(2n)]^\alpha$. Thus (1) holds for $n = 1, 2,...$ with $C = L[(b - a)/2]^\alpha$. The main point of t his note is the use of the following argument, employed in [1] and [2], to s how sufficiency. Let $a < x < y < b$. Let n_0 be the largest positive integer n for which $[x, y]$ lies in some $I_{n,j}$, $1 \leqslant j \leqslant n$. Then $y - x \geqslant (b - a)/(6n_0)$. For otherwise, if, say, $[x, y] \subset I_{n_0,j_0}$, $1 \leqslant j_0 \leqslant n_0$, then $[x, y]$ would lie either in one of the two (open) halves of I_{n_0,j_0} or in its (open) middle third; in each of these cases the maximality of n_0 is contradicted. By (1),

$$\begin{aligned} |f(y) - f(x)| &= |\{f(y) - s_{n_0}(y)\} + \{s_{n_0}(x) - f(x)\}| \\ &\leqslant 2C/n_0^\alpha \leqslant 2C[6/(b - a)]^\alpha (y - x)^\alpha. \end{aligned}$$

Remarks. 1. Observe that if a real function f satisfies in (a, b) a Lipschitz condition of order $\alpha(> 0)$, then it satisfies it in $[a, b]$, with appropriate values

$f(a), f(b)$. 2. If f is a real function on (a, b), and if for $n = 1, 2, \ldots$ there exists an $s_n \in S_n$ satisfying (1) with $\alpha > 1$, then by the Theorem, f must be constant in (a, b).

The above Theorem is given, under the explicit assumption that f is continuous in $[a, b]$, in the mimeographed notes [3]. Also, the Theorem is essentially the case $k = 0$ of Theorem 1 of [4].

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